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Yacine Chitour, Paolo Mason, Mario Sigalotti

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Abstract

Stability properties for continuous-time linear switched systems are at first determined by the (largest) Lyapunov exponent associated with the system, which is the analogous of the joint spectral radius for the discrete-time case. The purpose of this paper is to provide a characterization of marginally unstable systems, i.e., systems for which the Lyapunov exponent is equal to zero and there exists an unbounded trajectory, and to analyze the asymptotic behavior of their trajectories. Our main contribution consists in pointing out a resonance phenomenon associated with marginal instability. In the course of our study, we derive an upper bound of the state at time t, which is polynomial in t and whose degree is computed from the resonance structure of the system. We also derive analogous results for discrete-time linear switched systems.

1 Introduction

We consider linear switched systems of the form

$$\dot{x}(t) = A(t)x(t)\,, \tag{1}$$

where $x \in \mathbb{R}^n$ and the *switching law* $A(\cdot)$ is an arbitrary measurable function taking values on a compact and convex set $\mathcal{A}$ of $n \times n$ matrices. In the following, a switched system of the form (1) will be often identified with the corresponding set of matrices $\mathcal{A}$. This paper is concerned with stability issues for (1), where the stability properties are assumed to be uniform with respect to the switching law $A(\cdot)$.

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A characterization of the stability behavior of $\mathcal{A}$ relies on the sign of the *(largest) Lyapunov exponent* associated with $\mathcal{A}$, which is defined as

$$\rho(\mathcal{A}) = \sup \left(\limsup_{t \to +\infty} \frac{1}{t} \log \|x(t)\| \right), \tag{2}$$

where the sup is taken over the set of solutions of (1) with $\|x(0)\| = 1$ and $A(\cdot)$ an arbitrary switching law. The Lyapunov exponent is a "measure" of the asymptotic stability of (1). Indeed the system is (uniformly) exponentially stable if and only if $\rho(\mathcal{A}) < 0$. That means that there exist $C_1, C_2 > 0$ such that, for every trajectory of (1) with $A(\cdot)$ an arbitrary switching law, one has

$$\|x(t)\| \leq C_1 \exp(-C_2 t) \|x(0)\|, \quad t \geq 0.$$

On the other hand, (1) admits trajectories going to infinity exponentially fast if and only if $\rho(\mathcal{A}) > 0$. When $\rho(\mathcal{A}) = 0$, two situations may occur: (i) all trajectories of (1) starting from a bounded set remain uniformly bounded and there exist trajectories staying away from the origin, in which case (1) is said to be *marginally stable*; (ii) (1) admits a trajectory going to infinity and the system is said to be *marginally unstable*.

The role of the Lyapunov exponent is analogous to that of the joint spectral radius (or, equivalently, the generalized spectral radius) for discrete-time linear switched systems. The properties of the latter have been studied extensively in recent years (see for instance [5, 6, 7, 10, 16]). In particular, for discrete-time linear switched systems, several results have been obtained in the case in which the spectral radius is equal to one under particular assumptions (see for instance [11] and references therein). This case corresponds to the situation $\rho(\mathcal{A}) = 0$ for continuous-time systems of the form (1).

The stability properties of continuous-time systems in the case $\rho(\mathcal{A}) = 0$ have not attracted much attention in the community up to now. Some results, relating marginal stability of (1) to the existence of limit cycles and periodic trajectories can be found in [2, 4], while some general observations about marginal stability and instability can be found in [15]. It has to be noted that a qualitative study of the properties of the trajectories in the case $\rho(\mathcal{A}) = 0$ leads to similar properties for all values of ρ, since, as observed in [2], $\rho(\mathcal{A}') = 0$, where $\mathcal{A}'$ is the set $\{A - \rho(\mathcal{A})\mathrm{Id} \mid A \in \mathcal{A}\}$ with Id denoting the $n \times n$ identity matrix.

In this paper, we introduce the concept of resonance, which is rather natural for reducible switched systems with $\rho(\mathcal{A}) = 0$. Recall that if the switched system is marginally unstable then it must be reducible, i.e., there exist a linear change of coordinates transforming simultaneously every matrix of $\mathcal{A}$ into an upper block-diagonal matrix (see also Definition 1 below) and the sizes of the blocks do not depend on the particular matrix of $\mathcal{A}$. The diagonal blocks give rise to a finite number of switched systems of lower dimensions $\mathcal{A}_i$, $1 \leq i \leq r$, each of them being irreducible with nonpositive Lyapunov exponent.

We say that distinct subsystems $\mathcal{A}_i$ are in *resonance* if each of them is marginally stable and they admit trajectories staying away from the origin associated with the *same* switching law (cf. Definition 2). Our first result, Theorem 2, asserts that if a switched system is marginally unstable, then it admits subsystems in resonance. Clearly, the resonance phenomenon highlighted in Theorem 2 does not guarantee marginal instability (simply consider the case of block-diagonal matrices only). Understanding when marginal instability occurs seems a hard problem since it amounts to understand how the off-diagonal blocks interconnect the switched systems $\mathcal{A}_i$, $1 \leq i \leq r$, defined by the diagonal blocks of the elements of $\mathcal{A}$. We nevertheless address this issue in the case where $n \leq 4$ and $\mathcal{A}$ is the convex hull of a set of two matrices $\{A^0, A^1\}$. We show that there are no non-trivial examples satisfying the resonance hypothesis for $n = 2, 3$ and, for $n = 4$, we prove that, for almost every choice of the matrices A^0, A^1 satisfying the resonance hypothesis, (1) admits a trajectory going to infinity with linear growth as t tends to infinity.

We next pursue our study of the resonance phenomenon and we characterize an integer L, called the *resonance degree of* $\mathcal{A}$, which measures, in some sense, the complexity of the network of resonance relations between subsystems. In particular, the resonance degree is equal to zero when there is no resonance. We then state and prove our main result, Theorem 4, which provides an upper bound for the norm of the state at time t, which is polynomial in t of degree L and which is uniform with respect to the switching law $A(\cdot)$ defined on $[0, t]$. We extend the above study to discrete-time switched systems and obtain results entirely analogous to those dealing with the continuous-time case. In particular we improve an estimate provided in [14], which, to the best of our knowledge, was, up to now, the sharpest estimate on the polynomial behavior of trajectories of marginally unstable discrete-time switched systems.

We conclude the paper with several remarks regarding the worst asymptotic polynomial behavior, and with a comparison with the results and open questions in [11, 12]. In particular we estimate both the largest asymptotic growth of single trajectories as the time goes to infinity and the asymptotic growth, as t goes to infinity, of the largest norm of a point attainable at time t from the unit ball. We obtain, rather surprisingly, that, in some cases, the results of the two procedures do not coincide.

2 Resonance and marginally unstable switched systems

2.1 Notations and definition of resonance

In this section, we introduce the main notations of the paper as well as the concept of resonance for a reducible switched linear system $\mathcal{A}$ verifying $\rho(\mathcal{A}) = 0$. We also propose a first necessary condition for the marginal instability of a switched linear system.

If p is a positive integer, we use $\mathcal{M}_p(\mathbb{R})$ to denote the real vector space of $p \times p$ matrices with real entries.

Definition 1 We say that

$$\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{r-1} \subsetneq E_r = \mathbb{R}^n, \tag{3}$$

is an *invariant flag for (1) of length* r if every E_i is a subspace of $\mathbb{R}^n$ of dimension n_i which is invariant with respect to every matrix $A \in \mathcal{A}$. An invariant flag is said to be *maximal* if, for every $i = 1, \ldots, r$, there exists no subspace V such that $E_{i-1} \subsetneq V \subsetneq E_i$ and V is invariant with respect to $\mathcal{A}$. Finally an invariant flag is said to be *trivial* (resp. *nontrivial*) if $r = 1$ (resp. $r > 1$) and a switched system that admits (resp. does not admit) a nontrivial invariant flag is said to be *reducible* (resp. *irreducible*).

The following result relates the stability properties of a reducible switched system to those of lower dimensional irreducible switched systems.

Proposition 1 *Given a maximal invariant flag, there exists a vector basis $\{v_1, \ldots, v_n\}$ such that, for $i = 1, \ldots, r$, one has $E_i = \mathrm{span}\{v_1, \ldots, v_{n_i}\}$ and such that every matrix $A \in \mathcal{A}$ takes the following block form*

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & & \\ 0 & A_{22} & A_{23} & \cdots & \\ 0 & 0 & A_{33} & A_{34} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & A_{rr} \end{pmatrix}, \tag{4}$$

where each A_{ij} is an $(n_i - n_{i-1}) \times (n_j - n_{j-1})$ matrix with real entries. The subsystems of $\mathcal{A}$, defined as the switched systems corresponding to the sets $\mathcal{A}_i := \{A_{ii} \mid A \in \mathcal{A}\}$ for $i = 1, \ldots, r$, are irreducible and verify $\rho(\mathcal{A}_i) \leq \rho(\mathcal{A})$ for $1 \leq i \leq r$, with equality holding for at least one index i. Moreover, for a fixed switched system $\mathcal{A}$, the length r does not depend on the maximal flag and the subsystems $\mathcal{A}_i$ are unique up to reordering.

This proposition is a direct consequence of the Jordan-Hölder theorem for R-modules (see for instance [9, Theorem 13.7]). Note, moreover, that with any switching law $A(\cdot)$ in $\mathcal{A}$ it is naturally associated a switching law $A_{ii}(\cdot)$ in $\mathcal{A}_i$, for $i = 1, \ldots, r$.

From [2], we have the following fundamental result.

Theorem 1 *If (1) is irreducible, then there exists a norm $v(\cdot)$ in $\mathbb{R}^n$ such that, for every $x_0 \in \mathbb{R}^n$, one has*

(*a*) *for every trajectory $x(\cdot)$ of (1) starting from x_0, $v(x(t)) \leq v(x_0)e^{\rho(\mathcal{A})t}$, for every $t \geq 0$;*

(*b*) *there exists a trajectory $x(\cdot)$ of (1) starting from x_0 satisfying $v(x(t)) = v(x_0)e^{\rho(\mathcal{A})t}$, for every $t \geq 0$.*

A norm $v(\cdot)$ verifying condition (a) is called an *extremal norm*, while $v(\cdot)$ is called a *Barabanov norm* if it satisfies both (a) and (b) (see [16]).

Remark 1 An immediate consequence of the previous result is the nontrivial observation that an irreducible switched system (1) with $\rho(\mathcal{A}) = 0$ is stable and not asymptotically stable. Indeed, the balls with respect to a Barabanov norm are invariant for (1). On the other hand, for every initial condition x_0, there exists a trajectory of (1) lying on the *sphere* $v^{-1}(v(x_0))$, hence not converging to 0.

Remark 2 Let $\rho(\mathcal{A}) = 0$. Combining Proposition 1 with Theorem 1 and by a simple application of the variation of constant formula, we get that a trajectory can go to infinity at most polynomially. More precisely, there exists $C > 0$ such that, for every trajectory of (1) one has

$$\|x(t)\| \leq C(1 + t^{\hat{r}-1})\|x(0)\|, \quad t \geq 0, \tag{5}$$

where $\hat{r}$ is the number of irreducible subsystems associated with $\mathcal{A}$ which are stable and not asymptotically stable. This bound will be sharpened in Section 4.

We now introduce the notion of resonance between irreducible subsystems with zero Lyapunov exponent.

Definition 2 Consider a reducible switched system $\mathcal{A}$ and denote by $\mathcal{A}_1, \dots, \mathcal{A}_r$ the subsystems corresponding to a maximal invariant flag, as in Proposition 1. We say that the subsystems $\mathcal{A}_{i_1}, \dots, \mathcal{A}_{i_s}$ of $\mathcal{A}$, where $i_1, \dots, i_s$ are distinct indices, are *in resonance* if they satisfy the following conditions,

(a) $\rho(\mathcal{A}_{i_1}) = \cdots = \rho(\mathcal{A}_{i_s}) = 0$;

(b) there exists a switching law $A(\cdot)$ in $\mathcal{A}$ with associated switching laws $A_{i_j i_j}(\cdot)$ in $\mathcal{A}_{i_j}$ and corresponding trajectories $\gamma_{i_j}(\cdot)$ of $\mathcal{A}_{i_j}$ such that, for every $t > 0$ and for $j = 1, \dots, s$, $\gamma_{i_j}(t)$ belongs to the sphere $v_{i_j}^{-1}(1)$, where v_{i_j} is a Barabanov norm associated with $\mathcal{A}_{i_j}$.

The concept of resonance can be extended to any reducible switched system and does not actually depend on the specific Barabanov norms appearing in condition (b) in the above definition. The latter fact is a consequence of the following lemma.

Lemma 1 *We use the notations of Definition 2. Assume that the subsystems $\mathcal{A}_{i_1}, \dots, \mathcal{A}_{i_s}$ of $\mathcal{A}$ are in resonance and let $w_{i_1}(\cdot), \dots, w_{i_s}(\cdot)$ be extremal norms associated with $\mathcal{A}_{i_1}, \dots, \mathcal{A}_{i_s}$. Then there exists a switching law $\tilde{A}(\cdot)$ in $\mathcal{A}$ with associated switching laws $\tilde{A}_{i_j i_j}(\cdot)$ in $\mathcal{A}_{i_j}$ and corresponding trajectories $\tilde{\gamma}_{i_j}(\cdot)$ of $\mathcal{A}_{i_j}$ such that, for every $t > 0$ and for $j = 1, \dots, s$, $\tilde{\gamma}_{i_j}(t)$ belongs to the sphere $w_{i_j}^{-1}(1)$.*

Proof: Let $A(\cdot)$ be as in condition (b) of Definition 2. Consider a sequence $(t_k)_{k\in\mathbb{N}}$ of positive times going to infinity and such that $\gamma_{i_j}(t_k)$ converges to some $\gamma_{i_j}^*$ for $j = 1, \dots, s$. Since $\mathcal{A}$ is convex the space $L^\infty([0, +\infty), \mathcal{A})$ is closed in the weak-* topology and, by Banach-Alaoglu theorem, there exists a weak-* limit of $A(t_k + \cdot)$ in $L^\infty([0, +\infty), \mathcal{A})$, i.e., up to a subsequence, $A(t_k + \cdot) \overset{w^*}{\rightharpoonup} A^*(\cdot)$ in $L^\infty([0, +\infty), \mathcal{A})$. In particular, for $j = 1, \dots, s$, $A_{i_j}(t_k + \cdot) \overset{w^*}{\rightharpoonup} A_{i_j}^*(\cdot)$ in $L^\infty([0, +\infty), \mathcal{A}_{i_j})$ and (see for instance [13]) $\gamma_{i_j}(t_k + \cdot)$ converges uniformly on compact time-intervals to a solution $\gamma_{i_j}^*(\cdot)$ of (1). Hence,

$$w_{i_j}(\gamma_{i_j}^*(t)) = \lim_{k\to\infty} w_{i_j}(\gamma_{i_j}(t_k + t)) = \lim_{k\to\infty} w_{i_j}(\gamma_{i_j}(t_k)) = w_{i_j}(\gamma_{i_j}^*(0)).$$

The thesis is then satisfied with $A^*(\cdot) = \tilde{A}(\cdot)$ and $\tilde{\gamma}_{i_j}(\cdot) = \gamma_{i_j}^*(\cdot)/w_{i_j}(\gamma_{i_j}^*(0))$. $\square$

2.2 Stability in the absence of resonance

We can now state the first nontrivial result related to the notion of resonance.

Theorem 2 *Let $\mathcal{A}$ be a convex compact subset of $\mathcal{M}_n(\mathbb{R})$. Assume that the linear switched system associated with $\mathcal{A}$ is marginally unstable. Then $\mathcal{A}$ is reducible and, for any maximal invariant flag, it admits two subsystems $\mathcal{A}_{i_1}$, $\mathcal{A}_{i_2}$ in resonance.*

We will prove the theorem by contradiction, i.e., by showing that, if there are no subsystems of $\mathcal{A}$ in resonance, then the system is stable. Fix a maximal invariant flag such that all the matrices of $\mathcal{A}$ are of the form (4) and let $x = (x_1, \dots, x_r)$ where $x_i \in \mathbb{R}^{n_i - n_{i-1}}$ for $i = 1, \dots, r$. Consider a switching law $A(\cdot) \in \mathcal{A}$ and let $R_i(t, \tau)$, for $\tau, t \in \mathbb{R}$, be the resolvent of the time-varying linear system $\dot{z}_i = A_{ii} z_i$, $z_i \in \mathbb{R}^{n_i - n_{i-1}}$, i.e., $z_i(t) = R_i(t, \tau) z_i(\tau)$.

In particular, we have $x_r(t) = R_r(t, 0) x_r(0)$ and, since

$$\begin{aligned}\dot{x}_{r-1}(t) &= A_{r-1,r-1}(t)\, x_{r-1}(t) + A_{r-1,r}(t)\, x_r(t) \\ &= A_{r-1,r-1}(t)\, x_{r-1}(t) + A_{r-1,r}(t)\, R_r(t,0)\, x_r(0)\,,\end{aligned}$$

by the variation of constant formula, we get

$$x_{r-1}(t) = R_{r-1}(t,0) x_{r-1}(0) + \int_0^t R_{r-1}(t,\tau) A_{r-1,r}(\tau)\, R_r(\tau, 0)\, x_r(0)\,.$$

Repeating recursively the previous computations, we get

$$x_i(t) = R_i(t,0) x_i(0) + \sum_{s=1}^{r-i} \sum_{i<i_1<\cdots<i_s\le r} I(t, i, i_1, \dots, i_s) x_{i_s}(0),$$

where the integral $I(t, i, i_1, \dots, i_s)$ is defined as

$$\int_{t\ge\tau_1\ge\cdots\ge\tau_s\ge 0} R_i(t,\tau_1) A_{i,i_1}(\tau_1)\, R_{i_1}(\tau_1,\tau_2) \cdots A_{i_{s-1},i_s}(\tau_s) R_{i_s}(\tau_s, 0)\, d\tau_1 \cdots d\tau_s. \tag{6}$$

We will prove the proposition using the estimate

$$\|x_i(t)\| \le \Big(\|R_i(t,0)\| + \sum_{s=1}^{r-i} \sum_{i<i_1<\cdots<i_s\le r} \|I(t, i, i_1, \dots, i_s)\| \Big) \|x(0)\|, \tag{7}$$

and by estimating each $\|I(t, i, i_1, \dots, i_s)\|$. We first introduce the following matrix norms. For $1 \le i \le r$ and any matrix $M \in \mathcal{M}_{n_i - n_{i-1}}(\mathbb{R})$, define

$$\|M\|_i := \max_{\substack{z\in\mathbb{R}^{n_i-n_{i-1}} \\ v_i(z)=1}} v_i(Mz)\,,$$

where v_i is a Barabanov norm associated with $\mathcal{A}_i$. Since two norms defined on a finite dimensional vector space are equivalent, there exists $K_i > 0$ such that $\|M\| \le K_i \|M\|_i$ for

$i = 1, \dots, k$, where $\|\cdot\|$ denotes the usual matrix norm. Moreover the norms $\|\cdot\|_i$ are submultiplicative, i.e., for every pair of matrices M_1, M_2 in $\mathcal{M}_{n_i - n_{i-1}}(\mathbb{R})$, one has $\|M_1\, M_2\|_i \leq \|M_1\|_i\, \|M_2\|_i$. Finally, by definition, we have $\|R_i(\tau_1, \tau_2)\|_i \leq e^{\rho(\mathcal{A}_i)(\tau_1 - \tau_2)}$ for every choice of the switching law and $\tau_2 < \tau_1$. Since $\mathcal{A}$ is compact, there exists $K > 0$ independent of the switching law such the following holds true,

$$
\begin{aligned}
&\| \textstyle\int_{t \geq \tau_1 \geq \cdots \geq \tau_s \geq 0} R_i(t, \tau_1) A_{i,i_1}(\tau_1)\, R_{i_1}(\tau_1, \tau_2) \cdots A_{i_{s-1}, i_s}(\tau_s) R_{i_s}(\tau_s, 0)\, d\tau_1 \cdots d\tau_s \| \\
&\qquad \leq \textstyle\int_{t \geq \tau_1 \geq \cdots \geq \tau_s \geq 0} \|R_i(t, \tau_1) A_{i,i_1}(\tau_1)\, R_{i_1}(\tau_1, \tau_2) \cdots A_{i_{s-1}, i_s}(\tau_s) R_{i_s}(\tau_s, 0)\|\, d\tau_1 \cdots d\tau_s \\
&\qquad \leq K \textstyle\int_{t \geq \tau_1 \geq \cdots \geq \tau_s \geq 0} \|R_i(t, \tau_1)\|_i\, \|R_{i_1}(\tau_1, \tau_2)\|_{i_1} \cdots \|R_{i_s}(\tau_s, 0)\|_{i_s}\, d\tau_1 \cdots d\tau_s.
\end{aligned}
$$

Given $T > 0$, if we use $[\cdot]$ to denote the integer part of a real number, there exists $K' \geq 1$ depending on T but not on m nor on the switching law, such that

$$
\begin{aligned}
&\int_{t \geq \tau_1 \geq \cdots \geq \tau_s \geq 0} \|R_i(t, \tau_1)\|_i\, \|R_{i_1}(\tau_1, \tau_2)\|_{i_1} \cdots \|R_{i_s}(\tau_s, 0)\|_{i_s}\, d\tau_1 \cdots d\tau_s \\
&\leq K' \textstyle\int_{[\frac{t}{T}]T + T \geq \tau_1 \geq \cdots \geq \tau_s \geq 0} \|R_i([\frac{t}{T}]\, T + T, [\frac{\tau_1}{T}]\, T)\|_i\, \|R_{i_1}([\frac{\tau_1}{T}]\, T, [\frac{\tau_2}{T}]\, T)\|_{i_1} \cdots \|R_{i_s}([\frac{\tau_s}{T}]\, T, 0)\|_{i_s}\, d\tau_1 \cdots d\tau_s \\
&\leq K' T^s \textstyle\sum_{0 \leq m_s \leq \cdots \leq m_0 = [\frac{t}{T}] + 1} \|R_i(m_0 T, m_1 T)\|_i\, \|R_{i_1}(m_1 T, m_2 T)\|_{i_1} \cdots \|R_{i_s}(m_s T, 0)\|_{i_s} \\
&\leq K' T^s \textstyle\sum_{0 \leq m_s \leq \cdots \leq m_0} \prod_{j=1}^{m_s} \|R_{i_s}(jT, (j-1)T)\|_{i_s} \cdots \prod_{j=m_2+1}^{m_1} \|R_{i_1}(jT, (j-1)T)\|_{i_1} \prod_{j=m_1+1}^{m_0} \|R_i(jT, (j-1)T)\|_i.
\end{aligned}
\tag{8}
$$

We now want to prove that the previous sum is uniformly bounded with respect to the choice of the switching law and independently of m, at least when T is large enough. To this purpose, we need two subsidiary results given next.

Lemma 2 *Assume that $\mathcal{A}$ does not have subsystems in resonance. Then, for T large enough, there exists $C \in (0,1)$ such that, for every pair of distinct indices (i,j) with $1 \leq i, j \leq r$ and for every switching law,*

$$\|R_i(T, 0)\|_i \|R_j(T, 0)\|_j \leq C\,. \tag{9}$$

Proof: If $\rho(\mathcal{A}_i) < 0$ or $\rho(\mathcal{A}_j) < 0$, then (9) is true for every $T > 0$ for some $C \in (0,1)$. Therefore let us suppose without loss of generality that $i = 1$, $j = 2$ and $\rho(\mathcal{A}_1) = \rho(\mathcal{A}_2) = 0$. Proceeding by contradiction, let us assume that there exist sequences of switching laws $A^{(n)}(\cdot)$, initial data $x_l^{(n)}(0)$ with $v_l(x_l^{(n)}(0)) = 1$ for $l = 1, 2$ and times $T^{(n)}$, with $\lim_{n \to \infty} T^{(n)} = \infty$ such that $v_l(x_l^{(n)}(T^{(n)})) > 1 - \frac{1}{n}$ for $l = 1, 2$, where $x_l^{(n)}(\cdot)$ is the solution of the switched system $\mathcal{A}_l$ corresponding to $A^{(n)}(\cdot)$. As recalled in the proof of Lemma 1, up to a subsequence, $A^{(n)}(\cdot) \overset{w^*}{\rightharpoonup} A^*(\cdot)$ in $L^\infty([0, +\infty), \mathcal{A})$, and $x_l^{(n)}(\cdot) \overset{L^\infty_{\mathrm{loc}}([0,\infty))}{\longrightarrow} x_l^*(\cdot)$ for $l = 1, 2$, where $x_l^*(\cdot)$ is a solution of the switched system $\mathcal{A}_l$ corresponding to $A^*(\cdot)$. In particular $v_l(x_l^*(t)) = 1$ for $t > 0$ and $l = 1, 2$, contradicting the hypothesis of the lemma. □

To prove that the sum (8) is uniformly bounded, we use the following combinatorial result.

Lemma 3 *Let $h \in \mathbb{N}$, $h > 1$. Let us define*

$$\Xi_m = \{k \in \mathbb{N}^h \mid k_1 \leq k_2 \leq \cdots \leq k_h \leq m\}.$$

Moreover, given a set of real numbers

$$\alpha = \{\alpha_i^l \in (0,1] \mid l = 1, \ldots, h+1\,,\ i \in \mathbb{N}\}$$

and $k = (k_1, \ldots, k_h) \in \Xi_m$, let us define

$$\alpha_k = (\prod_{i=1}^{k_1} \alpha_i^1)(\prod_{i=k_1+1}^{k_2} \alpha_i^2) \cdots (\prod_{i=k_h+1}^{m} \alpha_i^{h+1})\,,$$

and

$$S_m(\alpha) = \sum_{k \in \Xi_m} \alpha_k\,.$$

Then, for any fixed $C \in (0,1)$, there exists a constant L depending on C such that $S_m(\alpha) \leq L$ for every $m \in \mathbb{N}$ and for every set α of the previous form satisfying in addition $\alpha_i^j \alpha_i^l \leq C$ for every $j \neq l$ and every $i \in \mathbb{N}$.

Proof: Let $k^{(1)}, k^{(2)} \in \Xi_m$ and observe that

$$\alpha_{k^{(1)}} \alpha_{k^{(2)}} \leq C^{\max_{l=1,\ldots,h} |k_l^{(1)} - k_l^{(2)}|}. \tag{10}$$

Indeed, assume without loss of generality that $k_{l_*}^{(1)} < k_{l_*}^{(2)}$, where

$$|k_{l_*}^{(1)} - k_{l_*}^{(2)}| = \max_{l=1,\ldots,h} |k_l^{(1)} - k_l^{(2)}|.$$

Let i be an integer verifying $k_{l_*}^{(1)} < i \leq k_{l_*}^{(2)}$. If $\alpha_i^{j_1}$ and $\alpha_i^{j_2}$ are terms corresponding to the subscript i in the factorization of $\alpha_{k^{(1)}}$ and $\alpha_{k^{(2)}}$, respectively, it is then easy to see that $j_2 \leq l_* < j_1$, implying that $\alpha_i^{j_1} \alpha_i^{j_2} \leq C$ from the hypothesis of the lemma. Thus (10) follows.

For $q \in \mathbb{N}$, let us define the set

$$\mathcal{I}_q = \{k \in \Xi_m \mid \alpha_k > C^q\},$$

and observe that, if $k^{(1)}, k^{(2)} \in \mathcal{I}_q$, then $\alpha_{k^{(1)}} \alpha_{k^{(2)}} > C^{2q}$ so that, from (10), we deduce that

$$\max_{l=1,\ldots,h} |k_l^{(1)} - k_l^{(2)}| < 2q, \qquad \forall k^{(1)}, k^{(2)} \in \mathcal{I}_q\,.$$

In particular, the set $\mathcal{I}_q$ contains at most $(2q)^h$ elements. Since $\Xi_m = \cup_{q=1}^{\infty} (\mathcal{I}_q \setminus \mathcal{I}_{q-1})$, we get

$$S_m(\alpha) = \sum_{q=1}^{\infty} \sum_{\hat{k} \in \mathcal{I}_q \setminus \mathcal{I}_{q-1}} \alpha_{\hat{k}} \leq \sum_{q=1}^{\infty} (2q)^h C^{q-1} < +\infty\,.$$

The lemma is proved by setting $L = \sum_{q=1}^{\infty} (2q)^h C^{q-1}$. □

The proof of the theorem is then concluded in view of Lemma 2 and by applying Lemma 3 to the estimates in (7) and (8).

Remark 3 The combined results of Proposition 1 and Theorem 2 are in the same spirit as Lemma 1 in [15], where the author states that a marginally unstable system admits a (possibly non-maximal) proper invariant flag of length two identifying two subsystems $\mathcal{A}_1$, $\mathcal{A}_2$ with $\rho(\mathcal{A}_1) = \rho(\mathcal{A}_2) = 0$. It should be noticed that the conclusion in [15, Lemma 1] goes a bit further by stating that both $\mathcal{A}_1$ and $\mathcal{A}_2$ can be taken marginally stable. However the latter statement is not true in general. Indeed, consider the switched system given by

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

The only proper invariant subspaces of $\mathcal{A}$ are $\mathbb{R} \times \{(0,0)\}$ and $\mathbb{R}^2 \times \{(0)\}$. Hence, one of the two subsystems associated with $\mathcal{A}$ is marginally unstable.

3 Sufficient conditions for marginal instability in dimension less than or equal to four

A natural question arising from the results of the previous section is whether, or under which additional conditions, a switched system with $\rho(\mathcal{A}) = 0$ and which admits subsystems in resonance is marginally unstable. A simple observation is that if $\rho(\mathcal{A}) = 0$ and if we assume that there exists a vector basis such that each matrix of $\mathcal{A}$ can be put in the block form (4) with $A_{ij} = 0$ for $i < j$ then the switched system (1) is stable, independently of the existence or non-existence of subsystems in resonance. Indeed in this case, setting $x = (x_1, \dots, x_r)$ where $x_i \in \mathbb{R}^{n_i - n_{i+1}}$ for $i = 1, \dots, r$, the components x_i of a trajectory of (1) vary independently one from each other and the stability of the overall system is therefore guaranteed by the fact that $\rho(\mathcal{A}_i) \leq 0$ (see Proposition 1). The role of the interaction terms A_{ij} is therefore fundamental to possibly show the existence of trajectories going to infinity. In the general case, a complete analysis of the contribution of these interaction terms is definitely a hard issue to address. Therefore we will limit ourselves to the rather explicit case where $n = 4$ and $\mathcal{A}$ is the convex hull of two matrices A^0 and A^1, denoted by $\mathrm{co}\{A^0, A^1\}$.

To state some of our subsequent results, we need the following definition.

Definition 3 A marginally unstable switched system associated with a convex compact subset $\mathcal{A}$ of $\mathcal{M}_n(\mathbb{R})$ is said to be *polynomially unstable of degree l* if there exists a positive integer l and constants $C_1, C_2 > 0$ such that every solution $x(\cdot)$ of (1) verifies $\|x(t)\| \leq C_1(1 + t^l)\|x(0)\|$ and there exists a solution $\bar{x}(\cdot)$ of (1) satisfying $\|\bar{x}(t)\| \geq C_2 t^l \|\bar{x}(0)\|$ for every $t > 0$.

Let $\mathcal{A} = \mathrm{co}\{A^0, A^1\} \subset \mathcal{M}_n(\mathbb{R})$ be a marginally unstable system. We consider the particular case in which all the matrices $uA^0 + (1-u)A^1$, where $u \in [0,1]$, are Hurwitz.

As a consequence of Theorem 2, it turns out immediately that $n \geq 4$, since otherwise one of the two subsystems obtained by applying Theorem 2 would be of dimension one and its maximal Lyapunov exponent would be equal to zero, leading to one of the two matrices A^0, A^1 having 0 as eigenvalue.

Let us fix $n = 4$. Marginal instability can only occur if the following holds true: there is a maximal invariant flag (3) with $k = 2$, $n_1 = 2$ and, for the associated subsystems, $\rho(\mathcal{A}_1) = \rho(\mathcal{A}_2) = 0$. In particular, we can write the matrices A^0, A^1 in the block form

$$A^0 = \begin{pmatrix} A^0_{11} & A^0_{12} \\ 0 & A^0_{22} \end{pmatrix}, \quad A^1 = \begin{pmatrix} A^1_{11} & A^1_{12} \\ 0 & A^1_{22} \end{pmatrix}, \tag{11}$$

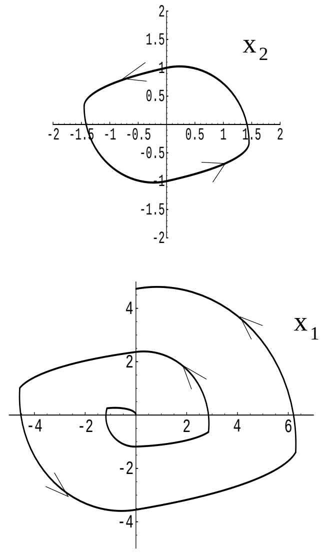

Figure 1: Trajectory of a polynomially unstable switched system (see Example 1).

where all the blocks are 2×2. We know from [1, 8] that the planar switched systems $\mathcal{A}_* = \mathrm{co}\{A_*^0, A_*^1\}$, with $\rho(\mathcal{A}_*) = 0$ and such that all the matrices of $\mathcal{A}_*$ are Hurwitz, are those admitting a closed worst trajectory. The latter corresponds to a periodic switching law $A_*(\cdot)$ obtained by concatenating a time-interval of length $t_0 > 0$ on which $A_*(t) = A_*^0$ with a second one of length $t_1 > 0$ on which $A_*(t) = A_*^1$ and then extending by periodicity. In the following, the times t_0, t_1 will be called *switching times*. It turns out that the period of the closed worst trajectory is equal to $2t_0 + 2t_1$, i.e., the worst trajectory is the concatenation of four *bang arcs*. For simplicity let us denote by T the half-period $t_0 + t_1$.

It is now easy to build an example of polynomially unstable switched system with matrices in the block form (11).

Example 1 *Assume that A^0, A^1 in (11) are such that $A_{11}^0 = A_{22}^0 = A_*^0$, $A_{11}^1 = A_{22}^1 = A_*^1$ and $A_{12}^0 = A_{12}^1 = \mathrm{Id}$, where $\mathcal{A}_* = \mathrm{co}\{A_*^0, A_*^1\}$ satisfies the properties detailed above. If the resolvent $R_*(t,0)$ corresponds to a worst switching strategy for $\mathcal{A}_*$ one can immediately verify, with the variation of constant formula, that*

$$\begin{aligned} x_2(t) &= R_*(t,0)x_2(0)\,, \\ x_1(t) &= R_*(t,0)x_1(0) + t\,R_*(t,0)x_2(0)\,, \end{aligned}$$

so that the system is polynomially unstable of degree 1. An explicit numerical example can be obtained by considering the matrices

$$A_*^0 = \begin{pmatrix} -1 & -\alpha \\ \alpha & -1 \end{pmatrix}, \qquad A_*^1 = \begin{pmatrix} -1 & -\alpha \\ 1/\alpha & -1 \end{pmatrix}.$$

For a value $\alpha \sim 4.5047$, one has $\rho(\mathcal{A}_*) = 0$ and Figure 1 depicts a particular trajectory for such a value.

In the following we consider closed worst trajectories of the form $t \mapsto R_*(t,0)\,\bar{x}_1$, for some $\bar{x}_1 \neq 0$, where the resolvent $R_*(t,0)$ corresponds to the switching law

$$A_*(t) = \begin{cases} A_*^0 & \text{if } t \in \big[kT, kT + t_0\big)\,, \ k \in \mathbb{N}, \\ A_*^1 & \text{if } t \in \big[kT + t_0, (k+1)T\big)\,, \ k \in \mathbb{N}. \end{cases} \tag{12}$$

It turns out that $\bar{x}_1$ is an eigenvector of the matrix $R_*(T,0)$ corresponding to the eigenvalue -1, while the modulus of the second eigenvalue is strictly smaller than one.

For switched systems $\mathcal{A} = \mathrm{co}(A^0, A^1)$, where A^0, A^1 are given by (11), Theorem 2 admits the following converse result (note that condition (C2) below is equivalent to requiring that $\mathcal{A}_1$ and $\mathcal{A}_2$ are in resonance).

Theorem 3 *Let A^0, A^1 be two 4×4 Hurwitz matrices in the block form* (11)*. We use $A_{12}(\cdot)$ to denote the top right 2×2 block in the switching law $A(\cdot)$. Assume that*

(C1) *the switched systems $\mathcal{A}_1 = \mathrm{co}\{A^0_{11}, A^1_{11}\}$ and $\mathcal{A}_2 = \mathrm{co}\{A^0_{22}, A^1_{22}\}$ admit closed worst trajectories with switching times t^1_0, t^1_1 and t^2_0, t^2_1, respectively,*

(C2) $t^1_0 = t^2_0 =: t_0$, $t^1_1 = t^2_1 =: t_1$,

(C3) *the condition*

$$\int_0^T \Pi_{x_1^1}\Big(R_1(T,\tau)A_{12}(\tau)R_2(\tau,0)\,x_1^2\Big)\,d\tau \neq 0$$

is satisfied. Here $R_k(\cdot,\cdot)$, $k = 1,2$, are the resolvents associated with the time-varying systems defining the worst trajectories, as in (12)*, and, similarly to what was done before, $T = t_0 + t_1$, x^k_1, x^k_2 are eigenvectors of $R_k(T,0)$, for $k = 1,2$, and $\Pi_{x_1^1}(x)$ is the component of the vector x along the direction x_1^1 with respect to the basis $\{x_1^1, x_2^1\}$.*

Then (1) *is polynomially unstable of degree one. Moreover assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ verify Conditions (C1) and (C2). Then, there exists a subset of pairs of matrices (A^0_{12}, A^1_{12}) which is open and dense in $\mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_2(\mathbb{R})$, such that Condition (C3) is verified.*

Proof: Let us consider the trajectory of (1) starting at $(0, x_1^2)^T$ and corresponding to the worst switching strategy for $\mathcal{A}_1, \mathcal{A}_2$,

$$x_2(t) = R_2(t,0)x_1^2 \tag{13}$$

$$x_1(t) = \int_0^t R_1(t,\tau)A_{12}(\tau)R_2(\tau,0)x_1^2 d\tau\,. \tag{14}$$

We first prove that the system is polynomially unstable under the hypotheses of the theorem. Fix $\tau \in [0,t]$ and consider the integer part $q(\tau) = \left[\frac{\tau}{T}\right]$ and the remainder $r(\tau) = \tau - T\left[\frac{\tau}{T}\right]$. Notice that

(I) the matrix $A_{12}(\cdot)$ only depends on $r(\tau)$, i.e.,

$$A_{12}(\tau) = A_{12}(r(\tau)),$$

(II) $R_1(\tau_1,\tau_2) = R_1(\tau_1 + T, \tau_2 + T)$ and $R_2(\tau_1,\tau_2) = R_2(\tau_1 + T, \tau_2 + T)$ for every $\tau_1, \tau_2 > 0$, since the period of the switching law is T,

(III) for all $0 \leq \tau \leq mT,\ m \in \mathbb{N}$, we can write

$$R_1(mT,\tau) = R_1\big(mT,(q(\tau)+1)T\big)\, R_1\big((q(\tau)+1)T,\tau\big)\,,$$
$$R_2(\tau,0) = R_2\big(\tau,q(\tau)T\big)\, R_2\big(T,0\big)^{q(\tau)}\,,$$

(IV) by definition of $\Pi_{x_1^1}$, if $v = v_1 x_1^1 + v_2 x_2^1 \in \mathbb{R}^2$, we have

$$\Pi_{x_1^1}\big(R_1(T,0)\,v\big) = v_1\,\Pi_{x_1^1}\big(R_1(T,0)\,x_1^1\big) + v_2\,\Pi_{x_1^1}\big(R_1(T,0)\,x_2^1\big) = -v_1 = -\Pi_{x_1^1}(v)\,.$$

Combining these facts we easily get

$$\Pi_{x_1^1}\Big(\int_0^{mT} R_1(mT,\tau)\,A_{12}(\tau)\,R_2(\tau,0)\,x_1^2\,d\tau\Big) = (-1)^{m+1}m\int_0^T \Pi_{x_1^1}\Big(R_1(T,\tau)\,A_{12}(\tau)\,R_2(\tau,0)\,x_1^2\Big)\,d\tau\,.$$

Then, under Condition (C3), $|\Pi_{x_1^1}(x_1(mT))| \geq C_1 m$, so that $\|x(mT)\| \geq C_2 m$ and, by the continuity of the resolvent of (1), we easily get $\|x(t)\| \geq C_3 t$, for suitable strictly positive constants C_1, C_2, C_3. The proof of the first part of the theorem is complete. We are left to prove that Condition (C3) is verified in an open and dense subset of $\mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_2(\mathbb{R})$ (in other words, that (C3) is verified generically with respect to the choice of the matrices (A_{12}^0, A_{12}^1)). This is a straightforward consequence of the following lemma.

Lemma 4 *With the notations above, let y be a nonzero vector in $\mathbb{R}^2$ orthogonal to x_2^1. Then, the linear map*

$$\begin{array}{rcl}\Psi\,:\,\mathcal{M}_2(\mathbb{R})\times\mathcal{M}_2(\mathbb{R}) & \to & \mathbb{R}\\ (A_{12}^0, A_{12}^1) & \mapsto & \displaystyle\int_0^T y^T R_1(T,\tau)\,A_{12}(\tau)\,R_2(\tau,0)\,x_1^2\,d\tau\end{array}$$

is onto.

Proof: Worst trajectories are such that, as τ varies in $[0,t_0]$, the vector $R_2(\tau,0)\,x_1^2$ lies in one of the four positive cones defined by the lines $\mathbb{R}x_1^2$ and $\mathbb{R}x_2^2$. Each of these cones is strictly contained in a half-plane. In particular, there exists $w \in \mathbb{R}^2$ such that $w^T R_2(\tau,0)\,x_1^2$ is strictly positive as τ varies in $[0,t_0]$.

Similarly, there exists $v \in \mathbb{R}^2$ such that $y^T R_1(T,\tau)\,v$ is strictly positive as τ varies in $[0,t_0]$.

The proof of the lemma is concluded by noticing that the pair $(A_{12}^0, A_{12}^1) = (vw^T, 0)$ has a nonzero image for the linear map Ψ. □

4 Worst polynomial behavior of marginally unstable switched systems

In this section, we generalize Theorem 2 and improve the relation (5) by providing a precise estimate of the maximal polynomial rate of divergence for a switched system on the basis of the resonance relations between subsystems. Thus, the purpose is to estimate the sum (8) in the presence of resonances. We start our analysis by introducing the following result, which sharpens Lemma 2. The notations are those of Section 2.2.

Lemma 5 *Fix a linear switched system $\mathcal{A}$ with $\rho(\mathcal{A}) = 0$ and a constant $\bar{C} \in (0,1)$. Then there exists $\bar{T} > 0$ such that the following holds. If, for any given switching law $A(\cdot)$, we set*

$$I(A(\cdot)) = \{i \in \{1,\dots,r\} \mid \|R_i(\bar{T},0)\|_i > \bar{C}\},$$

then the subsystems of $\mathcal{A}$ corresponding to the indices in $I(A(\cdot))$ are in resonance.

Proof: We first prove that there exist $\hat{T} > 0$ and $\hat{C} \in (0,1)$ such that, for any $A(\cdot)$, the subsystems of $\mathcal{A}$ corresponding to the indices in

$$\hat{I}(A(\cdot)) = \{i \in \{1,\dots,r\} \mid \|R_i(\hat{T},0)\|_i > \hat{C}\}$$

are in resonance.

By contradiction, assume there exist a sequence T_k converging to infinity, a sequence $C_k \in (0,1)$ converging to 1 and a switching law $A^{(k)}(\cdot)$ (whose resolvent is denoted by $R^{(k)}$) such that the subsystems corresponding to the indices in

$$I_k(A^{(k)}(\cdot)) := \{i \in \{1,\dots,r\} \mid \|R_i^{(k)}(T_k,0)\|_i > C_k\},$$

are not in resonance. Up to extracting a subsequence, we can assume that the set $I_k(A^{(k)}(\cdot))$ does not depend on k. The proof of the existence of $\hat{T} > 0$ and $\hat{C} \in (0,1)$ with the property stated above is then concluded by following that of Lemma 2: by extracting a weak limit of the sequence $\left(A^{(k)}(\cdot)\right)_{k\in\mathbb{N}}$, one shows the existence of a trajectory $x^*(\cdot)$ whose components $x_i^*(\cdot)$ satisfy $v_i(x_i^*(\cdot)) \equiv 1$ for all $i \in I_k(A^{(k)}(\cdot))$, contradicting the inductive hypothesis.

Let now $m \in \mathbb{N}$ be such that $\hat{C}^m \le \bar{C}$ and $M \le 2^r$ be the number of all subsets of $\{1,\dots,r\}$ whose corresponding subsystems are in resonance. Set $\bar{T} = (M(m-1)+1)\hat{T}$. The choice of $\bar{T}$ is such that, for every switching law $A(\cdot)$, the map associating with $j \in \{0,\dots,M(m-1)\}$ the set $\hat{I}(A(\cdot + j\hat{T}))$ takes at least m times the same value $F \subset \{1,\dots,r\}$. For any index $i \in \{1,\dots,r\}\setminus F$,

$$\|R_i(\bar{T},0)\|_i \le \Pi_{j=0}^{M(m-1)} \|R_i((j+1)\hat{T}, j\hat{T})\|_i \le \hat{C}^m \le \bar{C}.$$

Hence, $I(A(\cdot))$ is a subset of F, which implies that its indices are in resonance. □

The previous lemma motivates the introduction of some notations that will be useful in order to estimate the sum (8). The idea is that, taking $T = \bar{T}$ in (8), the values of $\|R_i(j\bar{T},(j-1)\bar{T})\|_i$ can be larger than $\bar{C}$ only on a set of indices i corresponding to subsystems of $\mathcal{A}$ which are in resonance.

Definition 4 A *chord* of height r is an r-dimensional vector whose components are all equal to 1 or $\bar{C}$. The *resonance degree* of a chord γ is the number of the components of γ that are equal to 1, minus 1. A *chord collection* $\mathcal{N}$ of height r is a finite collection of chords of identical height equal to r. A chord collection $\mathcal{N}$ is said to be *complete* if, given $\gamma \in \mathcal{N}$, for any other chord $\tilde{\gamma}$ with the property that all the $\bar{C}$-components of γ are also $\bar{C}$-components of $\tilde{\gamma}$, one has $\tilde{\gamma} \in \mathcal{N}$.

Two chords γ_1, γ_2 of a collection $\mathcal{N}$ *do not intersect* if the indices of the 1-components of γ_1 are all strictly larger (or smaller) than the indices of the 1-components of γ_2. Two chords γ_1, γ_2 of a collection $\mathcal{N}$ *intersect marginally* if the maximum (or minimum) of the indices of the 1-components of γ_1 coincides with the minimum (or maximum) of the indices of the 1-components of γ_2.

The *resonance degree* of a complete chord collection $\mathcal{N}$ is the maximum, among all subsets of $\mathcal{N}$ with at most marginal intersections of chords, of the sum of the resonance degrees of chords (see

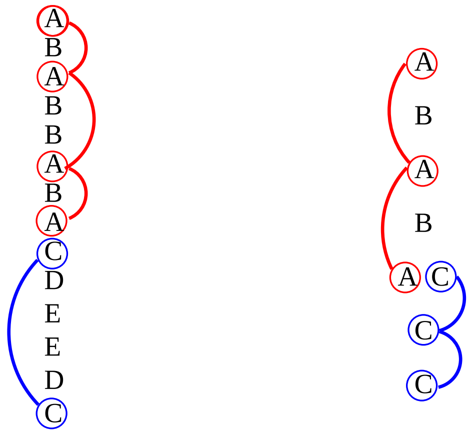

Figure 2: Two examples of chord collections with resonance degree equal to 4, using a compact notation; the components in resonance are represented by the same letter. Using the notations of this section, the chords of the collections are obtained by substituting 1 to components corresponding to the same letter and setting all the other components to $\bar{C}$.

Figure 2). More generally, the resonance degree of a chord collection $\mathcal{N}$ is defined as the resonance degree of the smallest complete chord collection containing $\mathcal{N}$.

The chords *upper* (resp. *lower*) *sub-collection* of height s of a collection $\mathcal{N}$ is the collection of the chords formed by the first (resp. last) s components of the chords of $\mathcal{N}$.

A *composition* $\mathcal{C}$ of length $m \leq \infty$ and height r of a given chord collection $\mathcal{N}$ of height r is a finite or infinite sequence of m chords of $\mathcal{N}$. Given a non-decreasing sequence $k = \{k_i\}_{i=1,\dots,m}$ of integers with $0 < k_i \leq r$, the *sample* σ_k of a composition $\mathcal{C}$ of length m is the sequence whose i-th element is the k_i-th component of the i-th chord. The *value* of a sample σ_k is the product of all its elements. The *value* $\mathcal{V}(\mathcal{C})$ *of a composition* $\mathcal{C}$ is the sum of the values of all its possible distinct samples. The *upper* (resp. *lower*) *sub-composition* of height s of a composition $\mathcal{C}$ is the composition obtained as the sequence of the chords formed by the first (resp. last) s components of the chords of $\mathcal{C}$.

In the previous definition, chords are intended to represent upper bounds on the values of the terms $\|R_i(j\bar{T},(j-1)\bar{T})\|_i$ in the sum (8) for fixed choices of the switching law. In particular, according to Lemma 5, resonances can be represented by a chord containing the corresponding components equal to 1, the other components being equal to $\bar{C}$. Thus, the value of a particular composition represents an estimate of the sum (8) in case some of the resonances are active on certain time-intervals, while the upper bound of all possible values of compositions provides a general upper bound for (8).

We are now ready to state the main technical result of this section, which generalizes Lemma 3.

Proposition 2 *Let $\mathcal{N}$ be a chord collection. Then there exists a constant $K > 0$ such that the value of all the possible compositions of $\mathcal{N}$ is bounded by $K(1+m^L)$, where L is the resonance degree of $\mathcal{N}$ and m is the length of the composition.*

Proof: Without loss of generality, we will assume that $\mathcal{N}$ is complete. First of all, let us observe that the proposition is true when $L = 0$ (Lemma 3). In the general case, the proof goes by induction on the height of the chord collection. It is clearly true when the height is equal to one

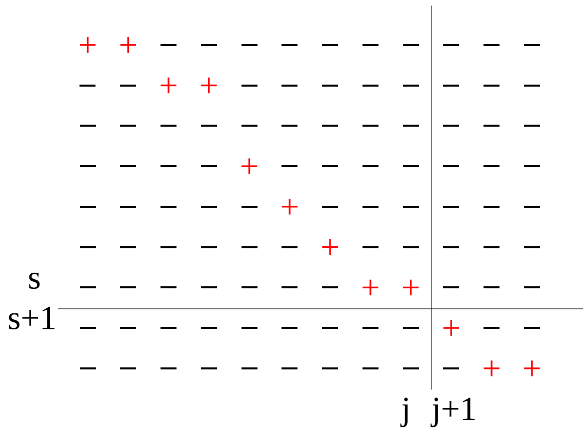

Figure 3: One possible choice of the sample (plus signs) as in the proof of Proposition 2.

(in particular there is no resonance in this case). Assume that the theorem is true for all complete chord collections of height at most $r \geq 1$. Take a complete chord collection of height $r+1$ with strictly positive resonance degree. Consider the largest $s > 0$ such that the resonance degree of the chord upper sub-collection of height s is equal to zero. Then we know that the value of the upper sub-composition of height s of any composition $\mathcal{C}$ is uniformly bounded by a constant $\hat{C}$. If $\hat{L} \geq 0$ is the resonance degree of the lower sub-collection of height $r+1-s$, we know by the inductive assumption that the value of the lower sub-composition of height $r+1-s$ of any composition $\mathcal{C}$ of length $\hat{m}$ is bounded by $\hat{K}(1+\hat{m}^{\hat{L}})$, for a suitable $\hat{K} \geq 0$. Fix now $j \in \{0, \dots, m\}$ and consider all the samples such that $k_j \leq s$ and $k_{j+1} > s$ (Figure 3). We deduce immediately from the previous estimates that the sum of the values of all these samples is bounded by $\hat{C}\hat{K}(1+(m-j)^{\hat{L}})$. By taking the sum over $j \in \{0, \dots, m\}$, we immediately obtain an estimate of the type

$$\mathcal{V}(\mathcal{C}) \leq K(1+m^{\hat{L}+1}),$$

for a suitable $K > 0$. Since, for some $\bar{s} \in \{1, \dots, s\}$, there exists a chord such that the $(s+1)$-th and the $\bar{s}$-th components are equal to one, we get that the resonance degree L of the chord collection satisfies $L \geq \hat{L}+1$.[1] The proposition is now proved. □

Our goal is to apply Proposition 2 to the case of marginally unstable switched systems. To this purpose, we introduce the following definition.

Definition 5 Let $\mathcal{A}$ be a convex compact subset of $\mathcal{M}_n(\mathbb{R})$ and consider an associated maximal invariant flag of length r. For any family $\{\mathcal{A}_{i_1}, \dots, \mathcal{A}_{i_s}\}$ of subsystems of $\mathcal{A}$ which are in resonance, consider the chord of length r with the $i_1, \dots, i_s$ components equal to 1 and all the other components equal to $\bar{C}$, where $\bar{C}$ belongs to $(0,1)$, and finally consider the corresponding (complete) chord collection. Then, the resonance degree of $\mathcal{A}$ is defined as the resonance degree of such chord collection.

[1] Actually "=" holds. Indeed, when passing from the collection of chords with non-intersecting resonances for the lower sub-collection to the global collection of chords with non-intersecting resonances, we have either to add the chord with just the $(s+1)$-th and $\bar{s}$-th components equal to one or to extend to the $\bar{s}$-th component a previous chord containing the $(s+1)$-th component. The reasoning works even in the case of marginally intersecting chords. In any case, the total resonance degree is obtained adding 1.

We are now able to state the main result of the section, Theorem 4 below, which is a straightforward consequence of Proposition 2, in view of (8).

Theorem 4 *Let $\mathcal{A}$ be a convex compact subset of $\mathcal{M}_n(\mathbb{R})$ with $\rho(\mathcal{A}) = 0$. Let L be the resonance degree of $\mathcal{A}$. Then there exists $\hat{K} \geq 1$ such that, for every solution of* (1)*, one has, for $t \geq 0$,*

$$\|x(t)\| \leq \hat{K}(1 + t^L)\|x(0)\|. \tag{15}$$

5 Worst polynomial behavior for discrete-time switched systems

In this section, we derive the counterpart of Theorem 4 for discrete-time linear switched systems. Consider

$$x(k+1) = A(k)x(k), \tag{16}$$

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$, and the switching law $A(k)$ takes values in a compact set of matrices $\mathcal{A} \subset \mathcal{M}_n(\mathbb{R})$. The *joint spectral radius* $\rho_d(\mathcal{A})$ is defined as

$$\rho_d(\mathcal{A}) := \lim_{k\to\infty} \max \left\{ \|A_{i_1} \cdots A_{i_k}\|^{1/k} \mid A_{i_1}, \dots, A_{i_k} \in \mathcal{A} \right\}. \tag{17}$$

Then system (16) is (uniformly) exponentially stable if and only if $\rho_d(\mathcal{A}) < 1$ and it admits trajectories going to infinity exponentially fast if and only if $\rho_d(\mathcal{A}) > 1$. When $\rho_d(\mathcal{A}) = 1$, two situations may occur: (*i*) all trajectories of (16) starting from a bounded set remain uniformly bounded and there exist trajectories staying away from the origin, in which case (16) is said to be *marginally stable*; (*ii*) (16) admits a trajectory going to infinity and the system is said to be *marginally unstable*.

Exactly as in the continuous-time case, one can associate with system (16) a maximal invariant flag and the corresponding block triangular matrix representations. The corresponding irreducible subsystems are unique up to reordering. Moreover, Barabanov norms exist for irreducible systems with $\rho_d(\mathcal{A}) = 1$ (cf. [3] and [16]). As in Remark 2, one has, for general discrete-time systems (16) verifying $\rho_d(\mathcal{A}) = 1$, that trajectories go (possibly) to infinity at most polynomially, according to the estimate

$$\|x(t)\| \leq C(1 + t^{\hat{r}-1})\|x(0)\|, \quad t \geq 0, \tag{18}$$

where $\hat{r}$ is the number of irreducible subsystems associated with $\mathcal{A}$ with joint spectral radius equal to one. The latter result was actually obtained in [14].

The resonance concept defined in Definition 2 carries over for system (16) and does not depend on the choice of a Barabanov norm. To see that, one must use a standard diagonal procedure instead of Banach–Alaoglu theorem. Lemma 5 also holds for systems (16) with the obvious modifications, i.e., the time $\bar{T}$ is now a positive integer and the resolvent $R_i(m_1, m_2)$ between times m_1 and m_2 is replaced by its discrete counterpart, namely, $R_i(m_1, m_2) = A_{ii}(m_1 - 1)A_{ii}(m_1 - 2) \cdots A_{ii}(m_2)$. Based on the discrete-time counterpart of Lemma 5, we can associate with $\mathcal{A}$ a chord collection and its resonance degree L_d. We can now state the following counterpart of Theorem 4.

Theorem 5 *Let $\mathcal{A}$ be a compact subset of $\mathcal{M}_n(\mathbb{R})$ with $\rho_d(\mathcal{A}) = 1$. Let L_d be the resonance degree of $\mathcal{A}$ defined as explained above. Then, there exists $\hat{c} \geq 1$ such that, for every solution of* (16)*, one has, for $m \geq 1$,*

$$\|x(m)\| \leq \hat{c} m^{L_d} \|x(0)\|. \tag{19}$$

The result follows from Proposition 2 and by an estimate analogous to (8). To get the latter one should notice that, using the notation $x = (x_1, \ldots, x_r)$ as in Section 2.2,

$$x_i(m) = \sum_{l=i}^{r} \sum_{i \leq i_1 \leq \cdots \leq i_{m-1} \leq l} A_{i\, i_1}(m-1) A_{i_1 i_2}(m-2) \cdots A_{i_{m-1} l}(0) x_l(0),$$

and the proof relies on the estimate of the matrix norm of

$$\sum_{i \leq i_1 \leq \cdots \leq i_{m-1} \leq l} A_{i\, i_1}(m-1) A_{i_1 i_2}(m-2) \cdots A_{i_{m-1} l}(0).$$

Assume for simplicity that m is a multiple of some fixed positive integer M, i.e., $m = qM$, then

$$\begin{aligned} &\Big\| \sum_{i \leq i_1 \leq \cdots \leq i_{m-1} \leq l} A_{i\, i_1}(m-1) A_{i_1 i_2}(m-2) \cdots A_{i_{m-1} l}(0) \Big\| \\ &\leq \hat{C} \sum_{i \leq i_M \leq \cdots \leq i_{(q-1)M} \leq l} \|R_i(qM, (q-1)M)\|_i \|R_{i_M}((q-1)M, (q-2)M)\|_{i_M} \cdots \|R_{i_{(q-1)M}}(M, 0)\|_{i_{(q-1)M}}, \end{aligned}$$

where $\hat{C}$ depends on the choice of M. One then simply proceeds as in the continuous-time case.

6 Remarks on the asymptotic behavior of trajectories

It has been shown previously that, for marginally unstable switched systems, trajectories may diverge to infinity at most polynomially, according to the estimates (15) and (19). The aim of this section is to investigate if these estimates are actually sharp. To this purpose we say that the *rate of polynomial growth* of a marginally unstable switched system $\mathcal{A}$ is the number p satisfying

$$C_1 t^p \leq \max_{A(\cdot), \|x(0)\|=1} \|x(t)\| \leq C_2 (1+t)^p \tag{20}$$

for suitable positive constants C_1 and C_2 and for all $t \geq 0$.

In [12, Theorem 3] the rate of polynomial growth for marginally unstable discrete-time linear switched systems given by matrices with nonnegative integer entries has been completely characterized. This result can actually be interpreted at the light of our estimates, by observing that the number characterized in [12, Theorem 3], similarly to our resonance degree, takes into account the subsystems of the switched system which are in resonance and, additionally, the interconnection among them through suitable off-diagonal terms.

The problem of the existence of an integer p and positive constants C_1 and C_2 such that (20) holds true seems to be difficult to address in full generality for marginally unstable switched linear system (it has been posed as an open question in [11, 12]). However the following result, which states that the estimate of Proposition 2 is actually optimal, suggests that a candidate value for the rate of polynomial growth of a marginally unstable switched linear system is its resonance degree.

Proposition 3 *Given a complete chord collection, there exists $\hat{C} > 0$ such that, for any $M \in \mathbb{N}$, it is possible to construct a composition $\mathcal{C}$ of length M such that*

$$\mathcal{V}(\mathcal{C}) \geq \hat{C} M^L. \tag{21}$$

Proof. Based on the definition of resonance degree of a complete chord collection, consider the chords $\gamma_1, \ldots, \gamma_k$ such that the indices of the components equal to 1 for γ_i are larger than or equal to those for γ_j, for $i > j$, and whose resonance degrees add up to L. Let $m = \left[\frac{M}{k}\right]$ and consider the composition obtained by taking γ_1 in the first m chords, γ_2 in the second m chords, and so on up to γ_k from the $((k-1)m+1)$-th to the M-th chord. Let us count the samples of this composition with value 1: it is easy to see that their number is comparable to $m^{j_1} m^{j_2} \cdots m^{j_{k-1}} (M-(k-1)m)^{j_k}$, where j_i is the resonance degree of the chord γ_i. Thus we get (21). □

The previous result motivates the following conjecture.

Conjecture 1. The rate of polynomial growth p of a generic continuous-time switched system $\mathcal{A} = \mathrm{co}\{A^1, \ldots, A^k\}$ with $\rho(\mathcal{A}) = 0$ is well-defined and equal to the resonance degree of $\mathcal{A}$. The same conclusion holds for a generic discrete-time switched system $\mathcal{A} = \{A^1, \ldots, A^k\}$ with $\rho_d(\mathcal{A}) = 1$.

Here, genericity should be intended in the spirit of Theorem 3, i.e., the diagonal blocks A^l_{jj} are fixed for $l = 1, \ldots, k$, $j = 1, \ldots, r$, and there exists an open dense subset $\mathcal{O}$ of some suitably defined linear space so that (22) holds true if the vector made of the off-diagonal blocks of the A^i belongs to $\mathcal{O}$. It seems reasonable that arguments adapted from the proof of Theorem 3 and Proposition 3, defining a control strategy that activates the resonant subsystems at different times, should lead to a positive answer to Conjecture 1.

Up to now we have considered global estimates on the polynomial growth of the ensemble of all trajectories. We now focus on the asymptotic behavior of single trajectories. We start our discussion with the following result.

Proposition 4 *Let $\mathcal{N}$ be a complete chord collection. Then there exists a composition of $\mathcal{N}$ of infinite length such that, if $\mathcal{V}_m$ is the value of the composition restricted to time m, then*

$$\limsup_{m\to\infty} \frac{\log \mathcal{V}_m}{\log m} = L,$$

where L is the resonance degree of $\mathcal{N}$.

Proof: We explicitly construct the composition by concatenating compositions of increasing length of the form described in the proof of Proposition 3. We choose the lengths of these compositions recursively as follows. Assume that we have constructed the whole composition up to time m_i and we want to complete it up to time m_{i+1} by adding a composition of the form described in the proof of Proposition 3. The value of the composition up to time m_{i+1} will then be larger than $\bar{C}^{m_i} \hat{C} (m_{i+1} - m_i)^L$ and, for $\varepsilon > 0$, up to assuming m_{i+1} large enough, this is larger than $m_{i+1}^{L-\varepsilon}$. By letting ε go to zero as i goes to infinity, we have that

$$\lim_{i\to\infty} \frac{\log \mathcal{V}_{m_i}}{\log m_i} = L,$$

which concludes the proof of the proposition. □

The previous result motivates the following conjecture.

Conjecture 2. For a generic $\mathcal{A} = \text{co}\{A^1, \ldots, A^k\}$ with $\rho(\mathcal{A}) = 0$, there exists a trajectory $x(\cdot)$ of (1) such that

$$\limsup_{t\to\infty} \frac{\log \|x(t)\|}{\log t} = L. \tag{22}$$

Analogously, for a generic $\mathcal{A} = \{A^1, \ldots, A^k\}$ with $\rho_d(\mathcal{A}) = 1$ there exists a trajectory of (16) such that (22) holds.

Here again, genericity should be intended as in Conjecture 1.

The next result shows that, in some cases, the asymptotic behavior of $\mathcal{V}_m$ is actually not comparable with m^L for any composition of infinite length. (Note that this is not in contradiction with Propositions 3 and 4.)

Proposition 5 *Let $\mathcal{N}$ be a complete chord collection and $\bar{C} \in (0,1)$. Let i be the largest integer such that the upper sub-collection of $\mathcal{N}$ of height i has resonance degree 0 and assume that $i\bar{C} < 1$. Let $\mathcal{N}'$ be the collection made of all the chords in $\mathcal{N}$ with at least one of the first i components equal to 1. If the resonance degrees of $\mathcal{N}$ and $\mathcal{N}'$ are different then*

$$\lim_{m\to\infty} \frac{\mathcal{V}_m}{m^L} = 0, \tag{23}$$

for any composition of infinite length, where $\mathcal{V}_m$ is the value of the composition restricted to time m and L the resonance degree of $\mathcal{N}$. Conversely, if $i = 1$ and the resonance degrees of $\mathcal{N}$ and $\mathcal{N}'$ coincide, then (23) does not hold for some composition of infinite length.

Proof: Assume that the resonance degrees of $\mathcal{N}$ and $\mathcal{N}'$ are different. It is clear that, for a composition of infinite length containing a finite number of elements of $\mathcal{N}\setminus\mathcal{N}'$, the limit (23) holds. Thus, assume that the composition contains an infinite sequence of elements of $\mathcal{N}\setminus\mathcal{N}'$. Let n_m be the number of such elements up to time m. For $\varepsilon > 0$, take m large enough such that $(i\bar{C})^{n_m} \le \varepsilon$. We want to estimate $\mathcal{V}_T$, the value of the composition up to time $T > m$. For this purpose, we consider all the samples such that $k_j \le i$ and $k_{j+1} > i$. Since the $r-i$ lower sub-collection of $\mathcal{N}$ has resonance degree $L-1$, it is easy to see that the sum of the values of all these samples is bounded by $K'(i\bar{C})^{n_j}(1+(T-j)^{L-1})$ for some positive constant K'. By taking the sum over $j \in \{0, \ldots, T\}$, we get

$$\mathcal{V}_T \le \sum_{j=1}^{T} K'(i\bar{C})^{n_j}(1+(T-j)^{L-1}) \le \sum_{j=1}^{m} K'(1+(T-j)^{L-1}) + \varepsilon \sum_{j=m+1}^{T} K'(1+(T-j)^{L-1}),$$

which implies

$$\limsup_{T\to\infty} \frac{\mathcal{V}_T}{T^L} \le \varepsilon K',$$

and then yields (23).

Suppose now that $i = 1$ and that the resonance degrees of $\mathcal{N}$ and $\mathcal{N}'$ are equal. Without loss of generality, we can assume that the lower sub-collection of $\mathcal{N}'$ of height $r-1$ has at most marginal intersections. We consider the composition of infinite length constructed in the proof of Proposition 4 for this sub-collection and we estimate the value of the whole composition at the times m_i. Since the first component of all the chords of $\mathcal{N}'$ is equal to 1, it turns out that this value coincides with the sum over j of all the values of the sub-composition between times j and m_i. Since $m_i - m_{i-1}$ is much larger than m_{i-1}, it is easy to see that the values of the sub-compositions are of the order of m_i^{L-1} on a set of indices j whose cardinality has order m_i. Thus, for this composition, the estimate (23) does not hold. □

Remark 4 To exemplify the last part of the statement of Proposition 5, consider the case where there is just one resonance. Then, the resonance degrees of $\mathcal{N}$ and $\mathcal{N}'$ coincide but the limit in (23) (if it exists) could be different from zero. This can be easily seen by taking the composition of infinite length containing only the chord with maximal resonance degree. The argument holds whether i is equal to 1 or not. The general case where the resonance degrees of $\mathcal{N}$ and $\mathcal{N}'$ coincide and $i > 1$ seems more difficult to handle.

The above proposition implies the following result regarding trajectories of linear switched systems.

Corollary 1 *Consider a linear switched system of the type* (1) *or* (16) *such that its corresponding complete chord collection $\mathcal{N}$ verifies the conditions of Proposition 5, i.e., the resonance degrees of $\mathcal{N}$ and $\mathcal{N}'$ are different. Then, for every trajectory $x(\cdot)$, one has*

$$\lim_{t\to\infty} \frac{\|x(t)\|}{t^{\Lambda}} = 0, \tag{24}$$

where Λ is equal to L or L_d.

Note that the above result implies that the notion of polynomial instability of degree l introduced in Definition 3 and applied in Theorem 3 is actually too restrictive in the general case.

We conclude this section with an explicit construction of a linear switched system of the type (1) for which the rate of polynomial growth coincides with the resonance degree and both (22) and (24) hold true. For this purpose, we consider a 8-dimensional system of the form $\mathcal{M} = \mathrm{co}\{M_0^0, M_1^0, M_0^1, M_1^1\}$

$$M_\varepsilon^0 = \begin{pmatrix} A^0 & \mathrm{Id} & 0 & 0 \\ 0 & A^0 & \varepsilon\mathrm{Id} & 0 \\ 0 & 0 & B^0 & \mathrm{Id} \\ 0 & 0 & 0 & B^0 \end{pmatrix}, \quad M_\varepsilon^1 = \begin{pmatrix} A^1 & \mathrm{Id} & 0 & 0 \\ 0 & A^1 & \varepsilon\mathrm{Id} & 0 \\ 0 & 0 & B^1 & \mathrm{Id} \\ 0 & 0 & 0 & B^1 \end{pmatrix}, \quad \varepsilon = 0, 1,$$

where the switched systems $\mathcal{A} := \{A^0, A^1\}$ and $\mathcal{B} := \{B^0, B^1\}$ are irreducible and stable, i.e.,

$$\rho(\mathcal{A}) = \rho(\mathcal{B}) = 0,$$

and not in resonance. Then, the resonance degree L of the system is equal to 2. According to Corollary 1, (24) holds true, with $\Lambda = 2$.

We choose now appropriate systems $\mathcal{A}$ and $\mathcal{B}$ and we construct switching laws illustrating that (20) is verified with $p = 2$ and (22) holds. The construction is based on two simple lemmas.

Lemma 6 *There exist $c > 0$ and a choice of the systems $\mathcal{A}$, $\mathcal{B}$ with $\rho(\mathcal{A}) = \rho(\mathcal{B}) = 0$ and not in resonance, such that, for any unit vector $v \in \mathbb{R}^2$ there exists a switching law defined in $[0, T]$ whose corresponding resolvents $R_A(t, s)$, $R_B(t, s)$, associated with $\mathcal{A}$ and $\mathcal{B}$ respectively, satisfy*

(i) v is an eigenvector of $R_B(T, 0)$ corresponding to the eigenvalue 1,

(ii) $\|(\int_0^T R_A(T, s) R_B(s, 0) ds) v\| \geq c$.

Proof. To prove the lemma one first chooses a system $\mathcal{B}$ whose worst trajectories are closed periodic curves of period T turning around the origin (see Section 3 and [1] for details). For each unit vector v, one can thus define a switching law such that the corresponding trajectory of $\mathcal{B}$ starting from v is periodic of period T, so that the corresponding resolvent $R_B(t,s)$ satisfies (i). It is easy, for instance by slightly slowing down the dynamics of $\mathcal{B}$, to construct matrices A^0, A^1 arbitrarily close to B^0, B^1, and such that the irreducible system $\mathcal{A}$ satisfies $\rho(\mathcal{A}) = 0$, admits closed worst trajectories, and is not in resonance with $\mathcal{B}$ (i.e., Condition (C2) in Theorem 3 is not satisfied). Then $(\int_0^T R_A(T,s)R_B(s,0)ds)v$ can be assumed to be arbitrarily close to $(\int_0^T R_B(T,s)R_B(s,0)ds)v = TR_B(T,0)v = Tv$, uniformly with respect to the initial choice of v. Thus the lemma is proved for any $c < T$. □

Lemma 7 *Assume that $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Lemma 6. Then there exist positive $C, \hat{C} > 0$ such that the following holds. For any $\bar{x} \in \mathbb{R}^8$, $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ with $\bar{x}_i \in \mathbb{R}^2$ there exists a trajectory $x(\cdot)$ of $\mathcal{M}$ starting from $\bar{x}$ such that*

$$\|x_1(t)\| \leq C(1+t)\|\bar{x}\| \quad \text{and} \quad \limsup_{t\to\infty} \frac{\|x_2(t)\|}{t} \geq \hat{C}\|\bar{x}_4\|.$$

Proof: We consider here a switching law taking values in $\{M_1^0, M_1^1\}$ and satisfying the conditions (i) and (ii) of Lemma 6 with $v = \bar{x}_4$. The estimate in $\|x_1\|$ follows basically from Proposition 2, since we are taking into account only one resonance relation. Let us compute the component $x_2(t)$. We have

$$\begin{aligned} x_2(t) &= R_A(t,0)\bar{x}_2 + \int_0^t R_A(t,s)x_3(s)ds \\ &= R_A(t,0)\bar{x}_2 + \int_0^t R_A(t,s)R_B(s,0)\bar{x}_3 ds + \int_0^t sR_A(t,s)R_B(s,0)\bar{x}_4 ds. \end{aligned}$$

We notice that the first term in the sum above is bounded. The second term is also bounded since $\mathcal{A}$ and $\mathcal{B}$ are not in resonance and because of Theorem 2. Let us consider the third term at a time $t = mT$ for a positive integer m. We have

$$\begin{aligned} &\int_0^{mT} sR_A(mT,s)R_B(s,0)\bar{x}_4 ds = \sum_{k=0}^{m-1} \int_{kT}^{(k+1)T} sR_A(mT,s)R_B(s,0)\bar{x}_4 ds \\ &= \sum_{k=0}^{m-1} R_A(mT,(k+1)T) \int_{kT}^{(k+1)T} sR_A((k+1)T,s)R_B(s,kT)R_B(kT,0)\bar{x}_4 ds \\ &= \sum_{k=0}^{m-1} \Big[kTR_A(T,0)^{m-k-1} \int_0^T R_A(T,s)R_B(s,0)\bar{x}_4 ds \\ &\qquad + R_A(T,0)^{m-k-1} \int_0^T sR_A(T,s)R_B(s,0)\bar{x}_4 ds \Big]. \end{aligned}$$

Since the modulus of the eigenvalues of $R_A(T,0)$ is strictly smaller than one, the matrix $\sum_{k=0}^{m-1} R_A(T,0)^{m-k-1}$ is uniformly bounded with respect to m. It remains to estimate the sum $\sum_{k=0}^{m-1} kR_A(T,0)^{m-k-1}\bar{w}$, where $\bar{w} = \int_0^T R_A(T,s)R_B(s,0)\bar{x}_4 ds$ satisfies, by Lemma 6, $\|\bar{w}\| \geq c\|\bar{x}_4\|$.

By a direct computation we have

$$\sum_{k=0}^{m-1} kR_A(T,0)^{m-k-1} = m\big(\mathrm{Id} - R_A(T,0)\big)^{-1} + \big(R_A(T,0)^m - \mathrm{Id}\big)\big(\mathrm{Id} - R_A(T,0)\big)^{-2},$$

proving the lemma, since $\lim_{m\to+\infty} R_A(T,0)^m = 0$. $\square$

To prove (20) with $p = 2$ we consider the following family of trajectories starting from $\bar{x} = (0,0,0,\bar{x}_4) \in \mathbb{R}^8$ with $\|\bar{x}_4\| = 1$. Using the previous lemma, we can choose $x(\cdot)$ such that $\|x_1(mT)\| \le C(1+mT)$ and $\|x_2(mT)\| \ge \hat{C}mT$. We then consider a switching law taking values in $\{M_0^0, M_0^1\}$ and such that the corresponding trajectory of $\mathcal{A}$ starting from $x_2(mT)$ is periodic, and we denote by $\hat{R}_A(t,s)$ the corresponding resolvent. We have in particular that $\|x_1(t)\| = \|\hat{R}_A(t,mT)x_1(mT) + (t-mT)\hat{R}_A(t,mT)x_2(mT)\| \ge \hat{c}mT(t-mT)$ for a suitable constant $\hat{c} > 0$. Given any $t > 0$, we consider the construction above with $m = [\frac{t}{2T}]$ and we recover (20) with $p = 2$.

To construct a trajectory satisfying (22), we proceed as follows. We start from any $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) \in \mathbb{R}^8$ with $\bar{x}_4 \ne 0$. Similarly to the previous construction, we can choose $x(\cdot)$ starting from $\bar{x}$ so that $\|x_1(2t_1)\| = \|\hat{R}_A(2t_1,t_1)x_1(t_1) + t_1\hat{R}_A(2t_1,t_1)x_2(t_1)\| \ge \hat{c}\|\bar{x}_4\|t_1^2$ for a suitable constant $\hat{c} > 0$, if t_1 is appropriately chosen.

The procedure can then be iterated to obtain $x(T_{k+1})$ starting from $x(T_k)$, where $T_k = 2\sum_{i=1}^k t_i$, and the times t_i can be chosen in such a way that

$$\frac{\log\|x(T_{k+1})\|}{\log T_{k+1}} \ge \frac{\log(\hat{c}\|x_4(T_k)\|t_{k+1}^2)}{\log T_{k+1}}$$

converges to 2, so that (22) is satisfied.

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